Supermatrix Models*

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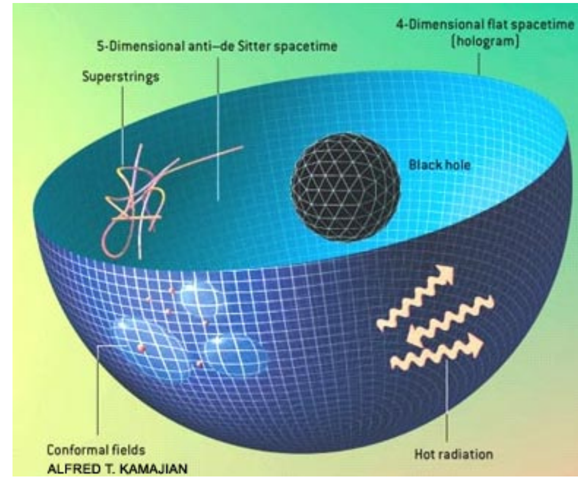
** Work with Cumrun Vafa and Ben Heidenreich, in progress*

Holography/Emergent Geometry

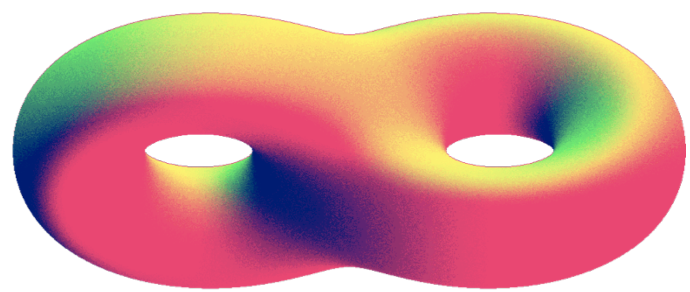
Quantum Theory
in d dimensions



d+1 Space-Time Gravity + QFT

Matrix Models (d=0)

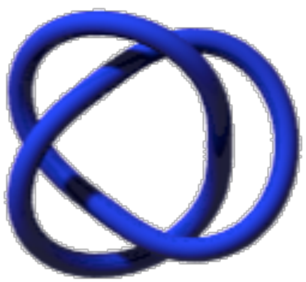
$$\int_{N\times N} d\Phi \cdot e^{\mathrm{Tr}\, W(\Phi)/g_s}$$



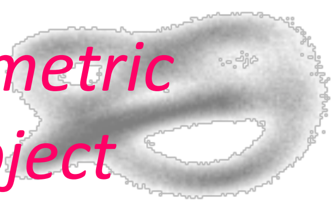
Quantum Spectral Curves
(Riemann Surface + CFT)

Quantization

Geometry



K



geometric object

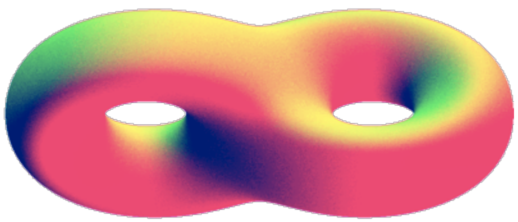
Algebra

$$Z(K) \in C$$

quantum invariant

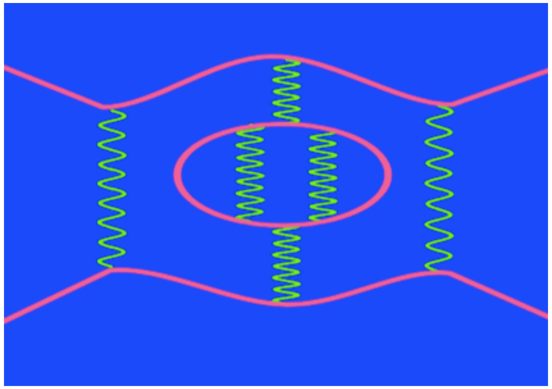
Emergence

Geometry



effective geometry

Algebra



quantum system

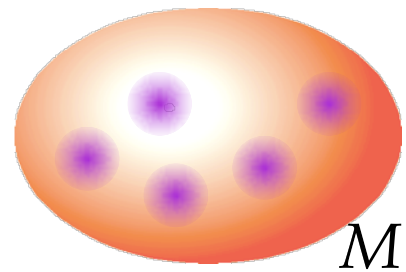
Supersymmetric Gauge Theory: N=4

Path integral localizes to instantons

$$Z = \int DA \cdot e^{-S} = \sum_{n \geq 0} d(n) q^n, \quad q = e^{2\pi i \tau}, \ \tau \text{ coupling}$$

Contribution of moduli space

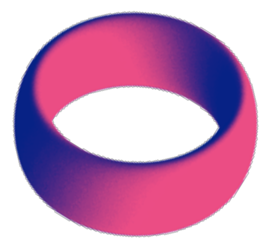
$$Mod_{N,n}(M)$$



M

SL(2,**Z**) S-duality in N=4 gauge theories ↔ modular invariance of a quantum CFT on 2-torus

$$\tau \to \frac{a\tau + b}{c\tau + d}$$



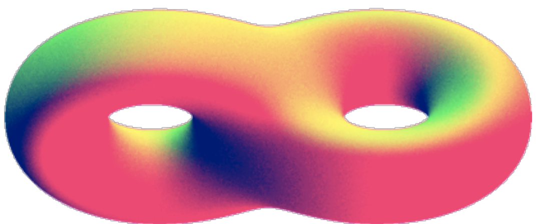
T^2

Related to 2d CFT characters

$$Z = \mathrm{Tr}\, q^{L_0} = \chi(q)$$

Supersymmetric Gauge Theory: N=2

Seiberg-Witten curve



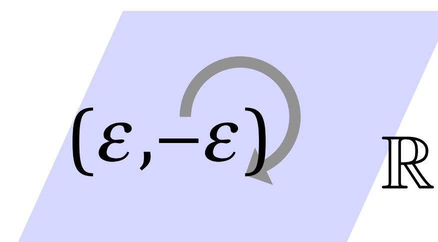
genus g

(Σ,ω), $\omega =$ meromorphic one-form

Effective gauge coupling $\tau_{ij}(x)$, BPS masses $\oint_C \omega$

Nekrasov deformation

$$Z = e^F, \quad F = \sum_g \varepsilon^{2g-2} F_g,$$

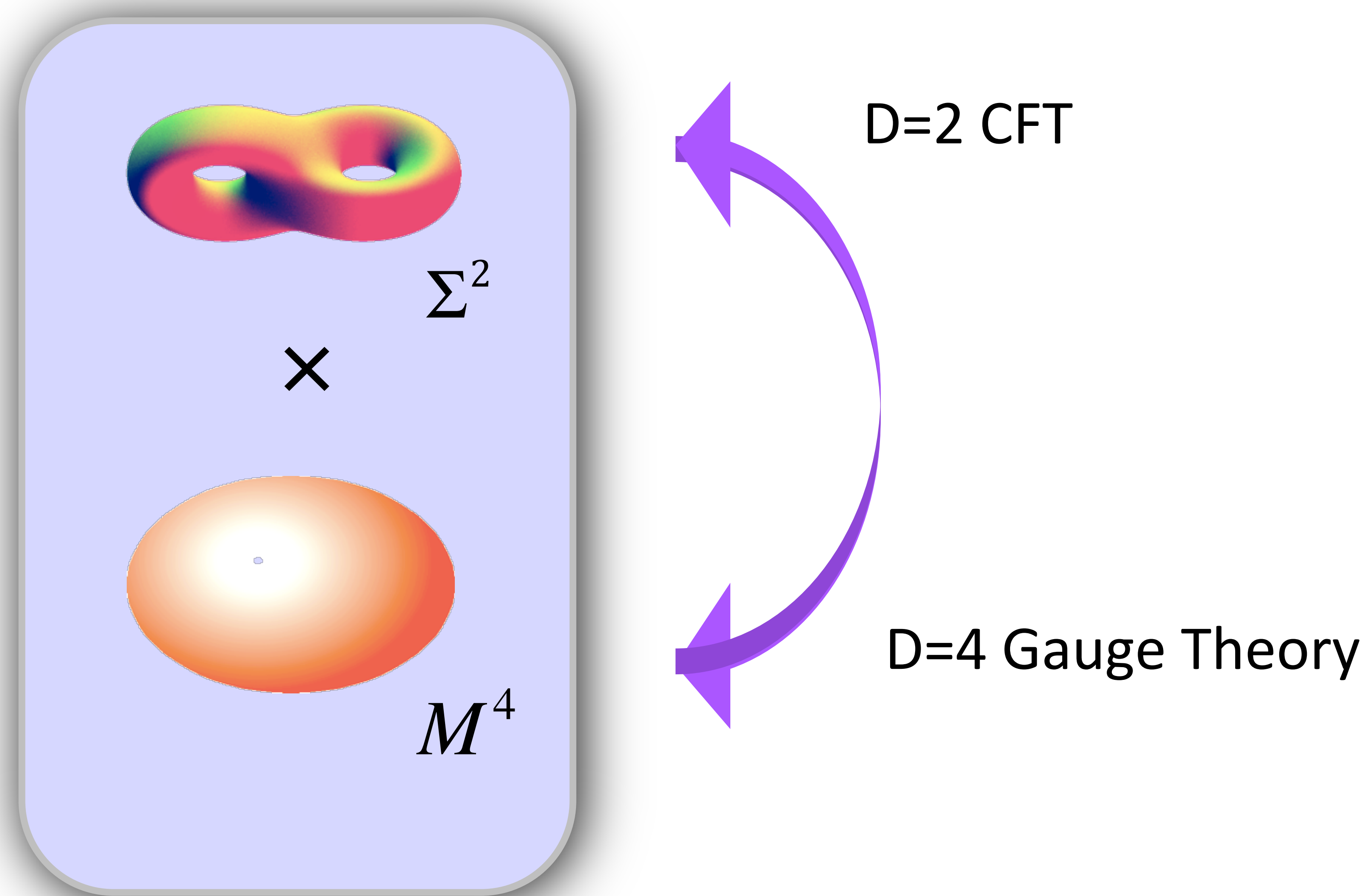


$\mathbb{R}^4 \cong \mathbb{C}^2$

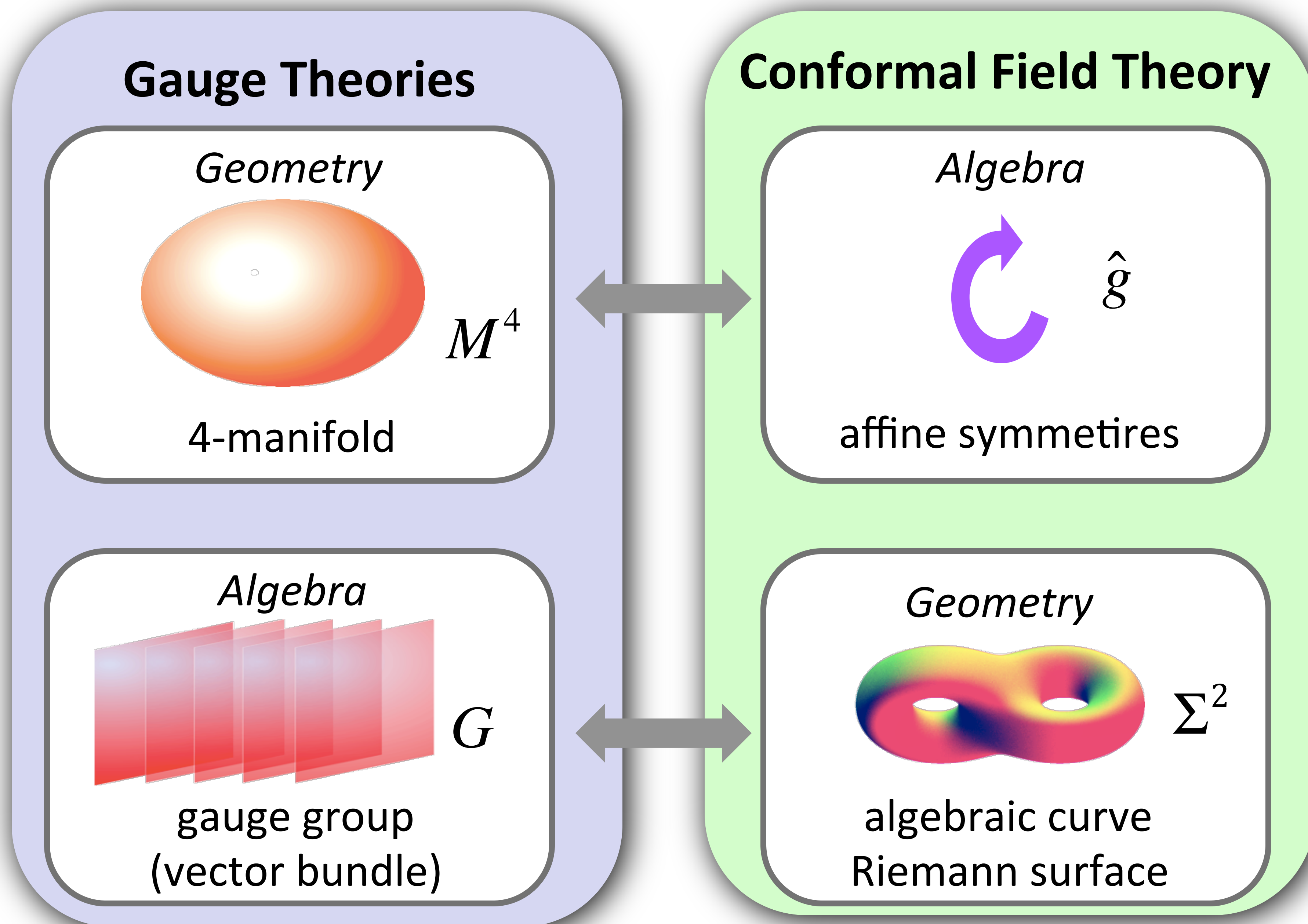
$$F_0: \quad \mu_i = \int_{A_i} \omega, \quad \frac{\partial F_0}{\partial \mu_i} = \int_{B_i} \omega$$

$$F_1 = \frac{1}{2} \log \det \Delta$$

D=6 (2,0) Tensor (2-form) SCFT



Geometry/Algebra Duality

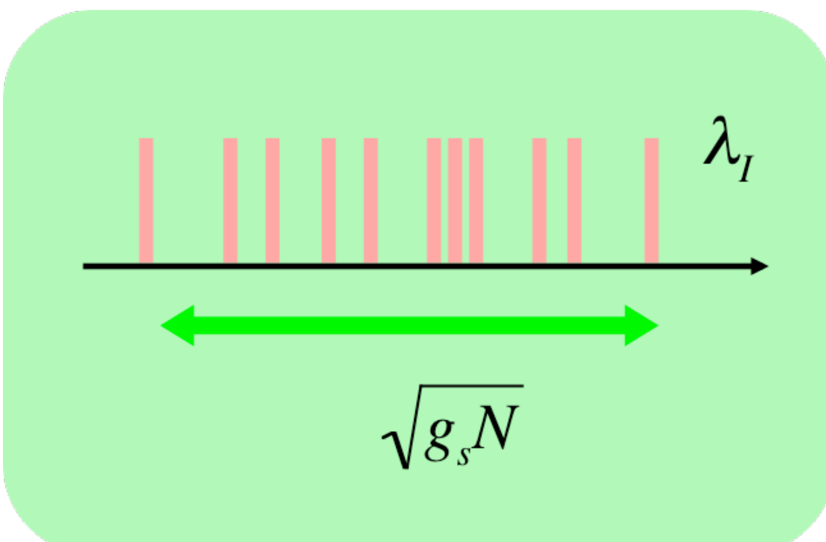


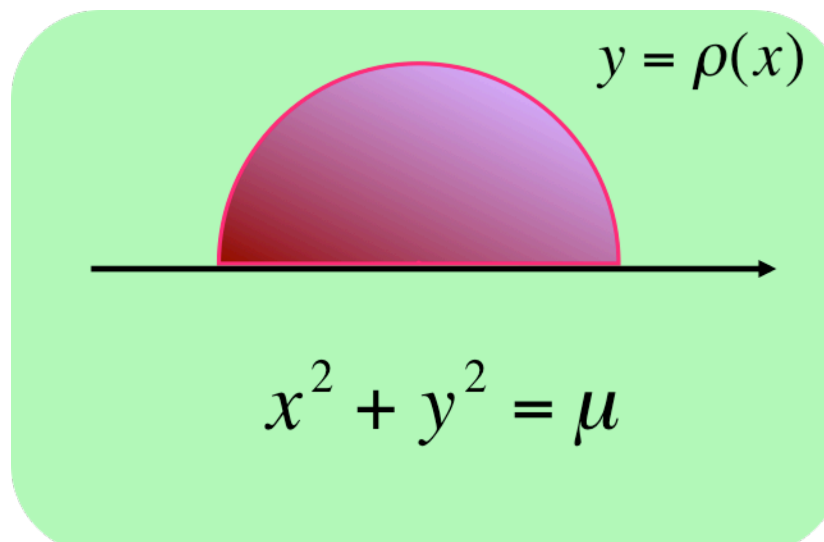
Wigner's Random Matrix Model

$$\lim_{N\to\infty} Z_N, \quad Z_N = \frac{1}{\mathrm{Vol}\, U(N)} \int_{N\times N} d\Phi \cdot e^{-\mathrm{Tr}\,\Phi^2/g_s}$$

Eigenvalue distribution in 't Hooft limit

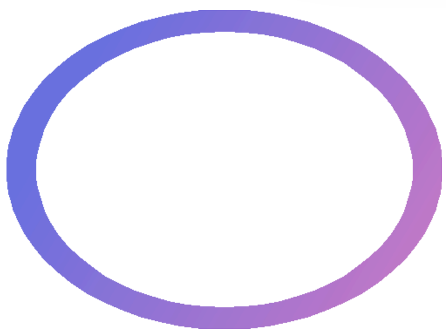
$$N \to \infty, \quad g_s \to 0, \quad N g_s = \mu = fixed$$



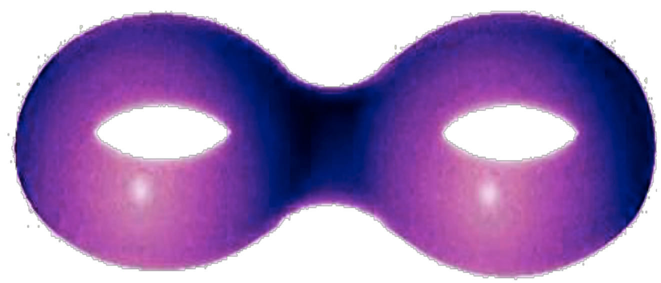


String Partition Function

$$Z = \exp \sum_{g \geq 0} g_s^{2g-2} F_g(\mu)$$



ribbon diagrams
1/N expansion



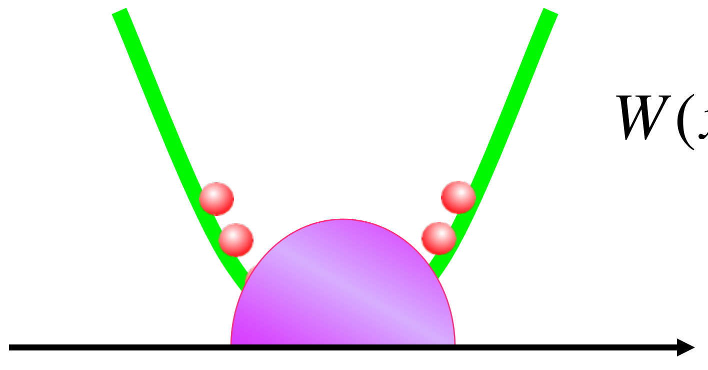
genus g
string surface

Eigenvalue Dynamics

$$Z_{matrix} = \int d^N\lambda \cdot \prod (\lambda_I - \lambda_J)^2 \cdot e^{-\sum_I \lambda_I^2 / g_s}$$

Effective action (repulsive Coulomb force)

$$S_{eff} = \sum_I \lambda_I^2 - 2g_s \sum_{I<J} \log(\lambda_I - \lambda_J)$$



$$W(x) = x^2$$

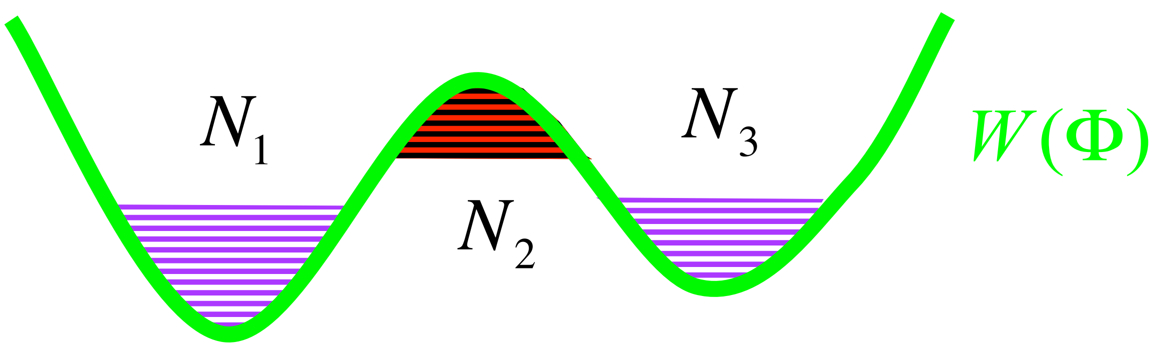
General Matrix Model

$$Z_{matrix} = \frac{1}{\mathrm{Vol}\, U(N)} \int_{N\times N} d\Phi \cdot e^{\mathrm{Tr}\, W(\Phi)/g_s}$$

’t Hooft limit

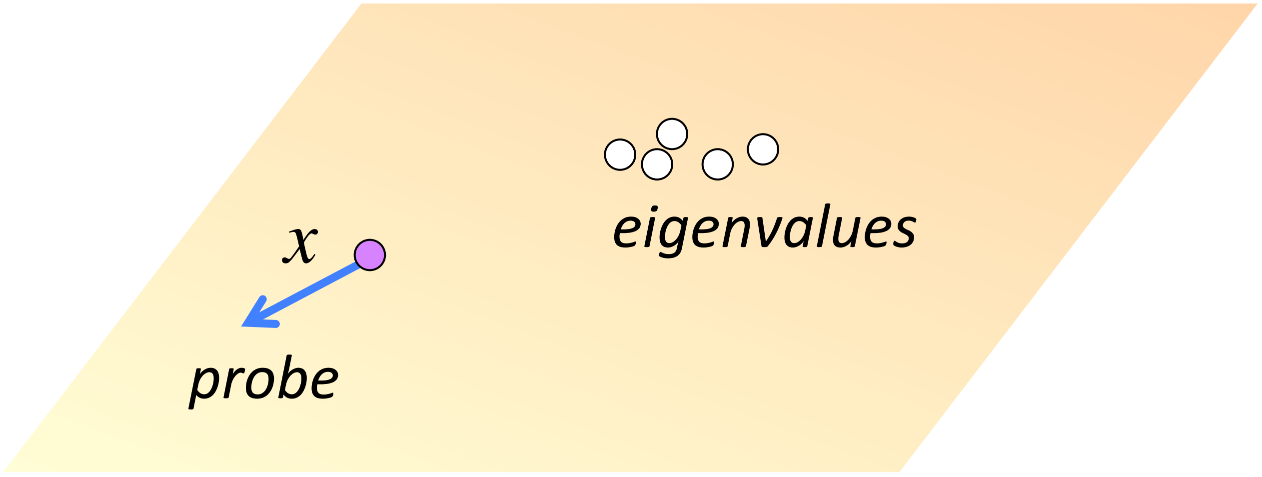
$$N_I \to \infty, \quad g_s \to 0, \quad N_I g_s = \mu_I = fixed$$

Filling fractions (perturbative expansion)



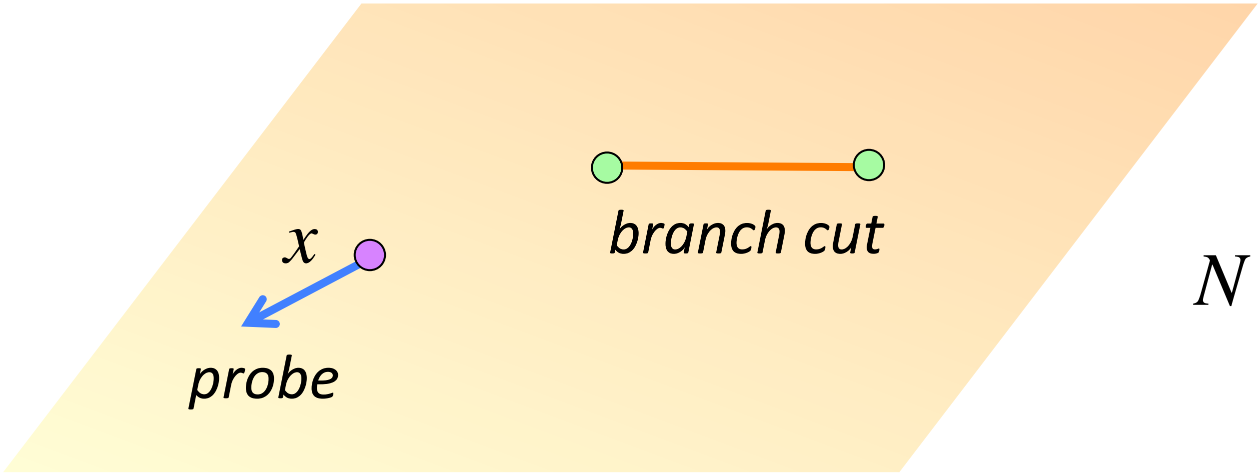
Resolvent

$$y\,dx = dS_{eff} = dW(x) + 2g_s \mathrm{Tr}\frac{dx}{\Phi - x}$$



Resolvent

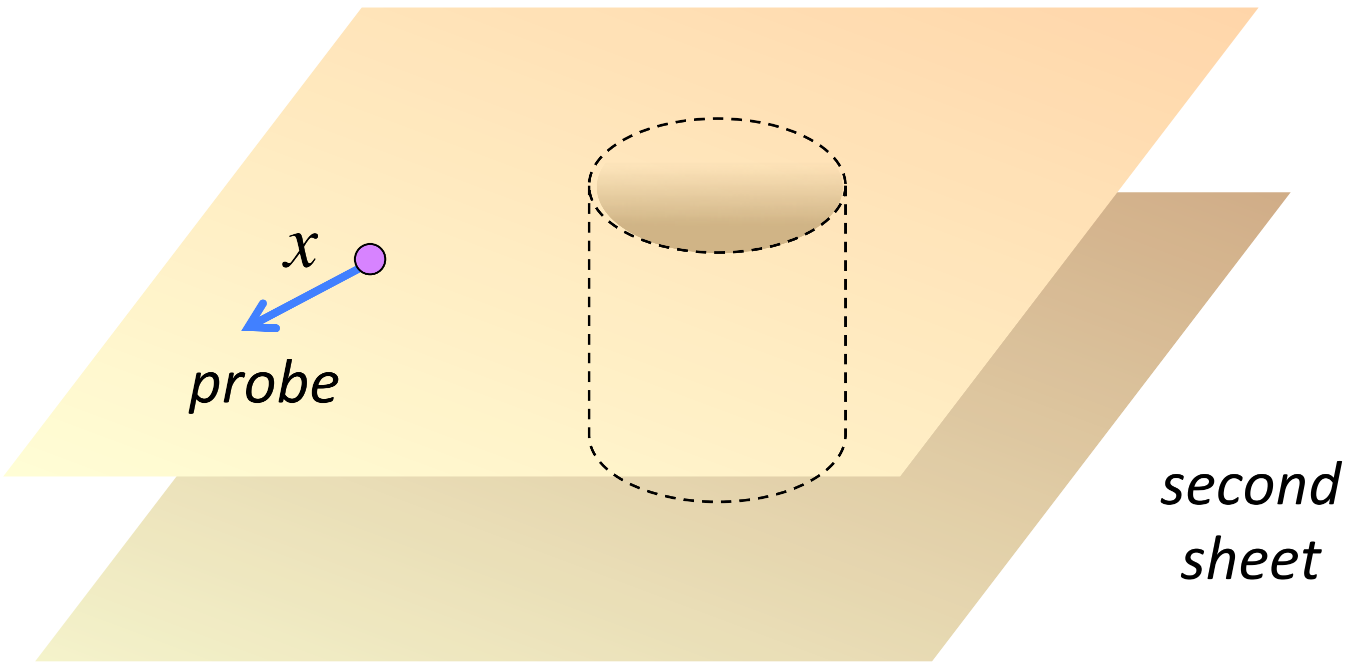
$$y\,dx = dS_{eff} = dW(x) + 2g_s \,\mathrm{Tr}\frac{dx}{\Phi - x}$$



$N \to \infty$

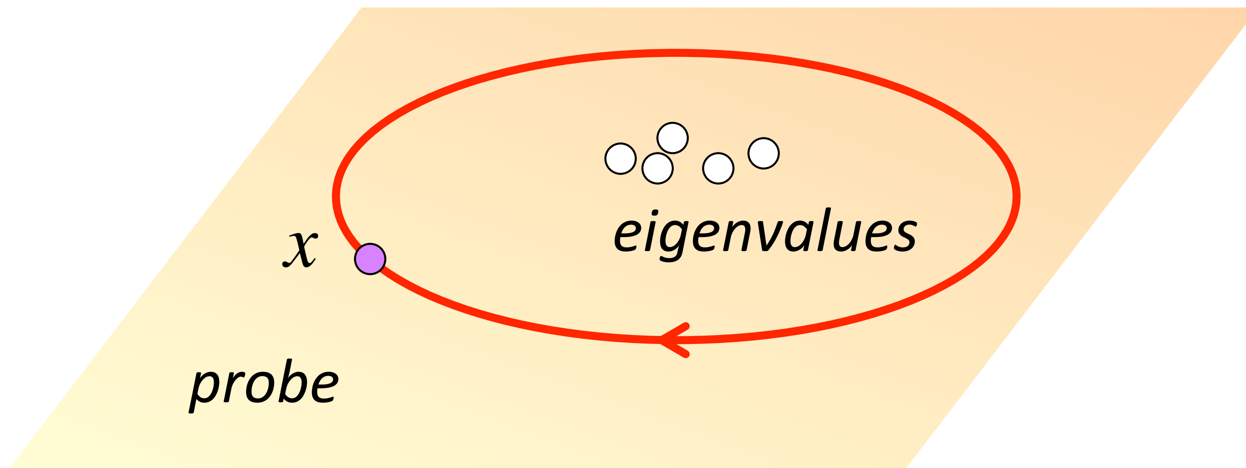
Effective Geometry

$$y^2 = W_n'(x)^2 + f_{n-1}(x) = P_{2n}(x)$$



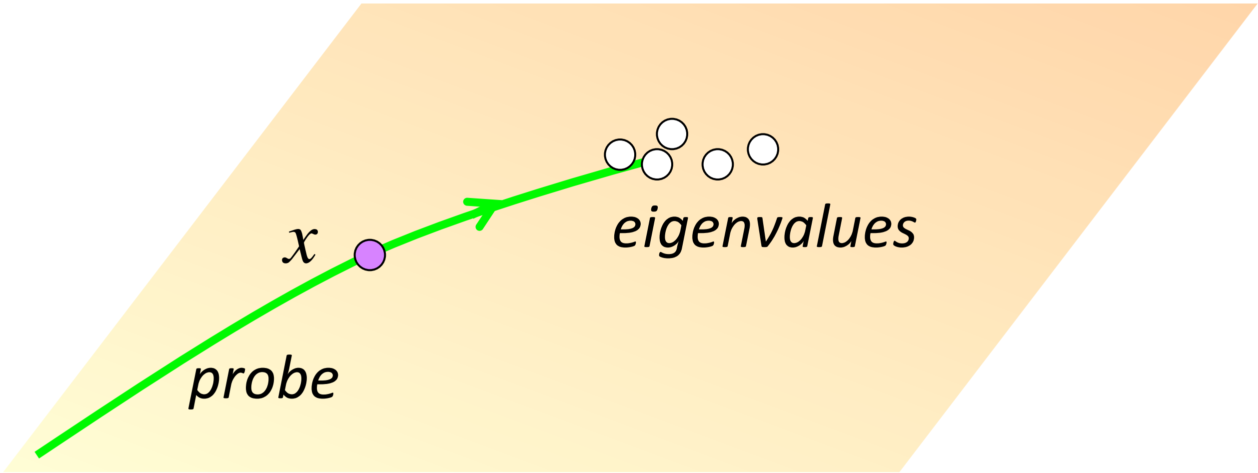
Periods (A)

$$\oint_{A_I} y\,dx = g_s N_I = \mu_I$$



Periods (B)

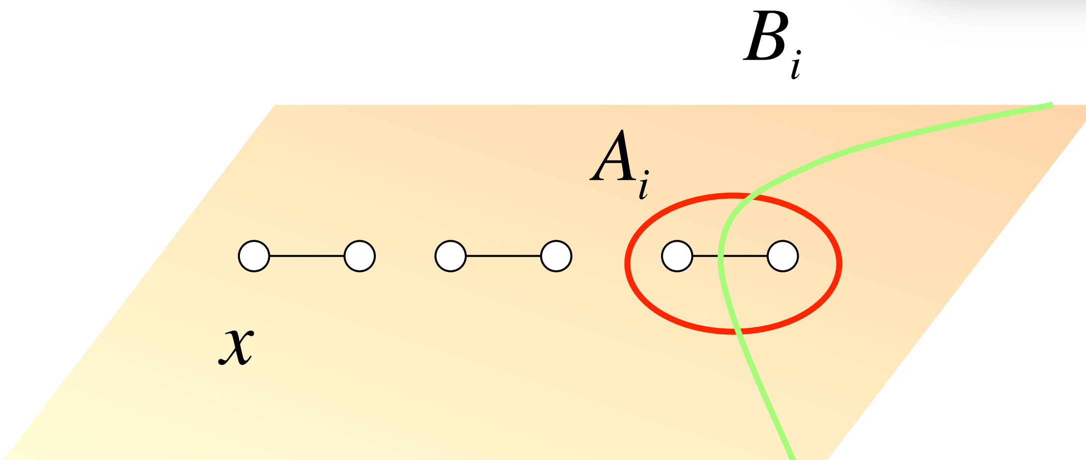
$$\int_{B_I} y\,dx = \frac{\partial F}{\partial \mu_I}$$

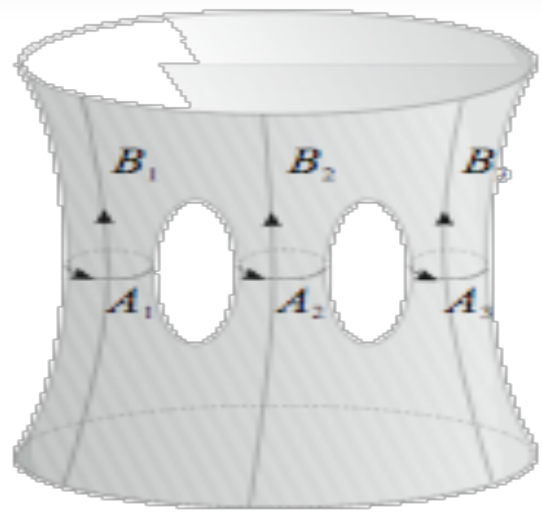


Spectral Curve

Hyperelliptic curve

$$\Sigma : y^2 = W'(x)^2 + f(x)$$





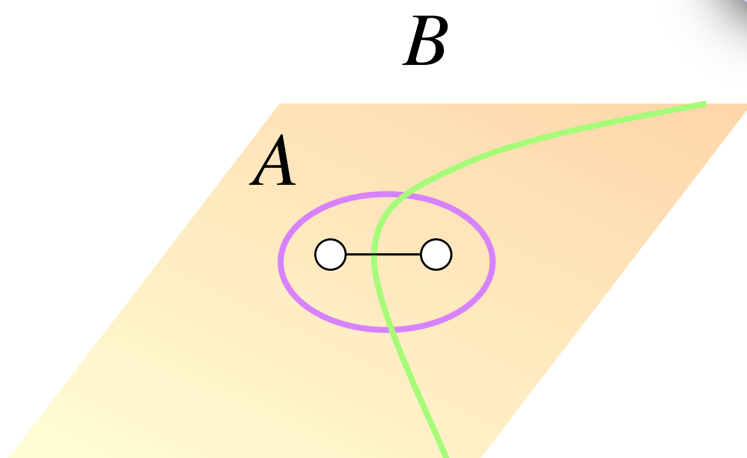
Quantum invariants of $\Sigma, \quad \omega = ydx$

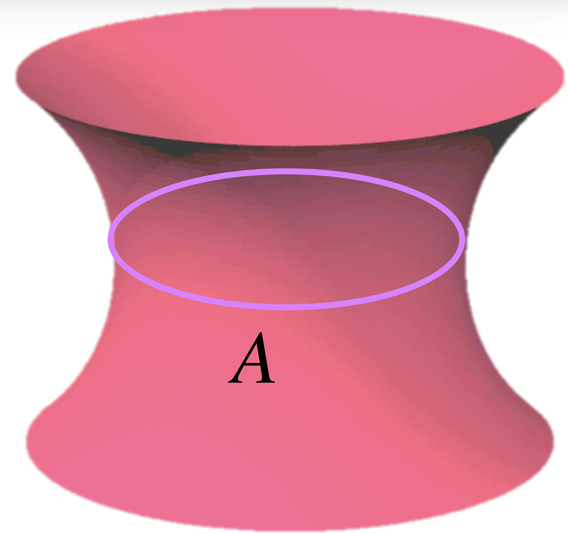
$$\mu_i = \oint_{A_i} ydx \qquad \frac{\partial F_0}{\partial \mu_i} = \oint_{B_i} ydx$$

Wigner's Semi-Circle

Rational spectral curve

$$\Sigma : y^2 = x^2 + \mu$$

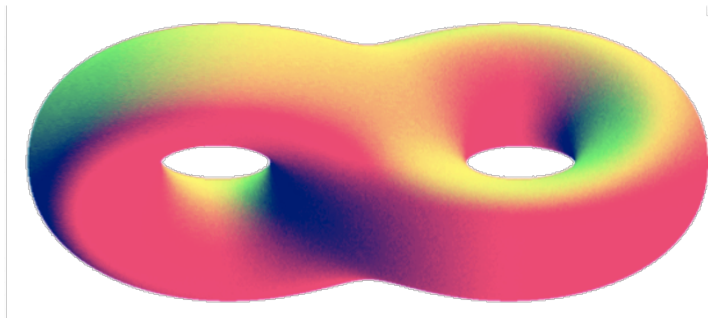




$$F_0 = \frac{1}{2}\mu^2 \log \mu = -\log \; Vol\big(U(N)\big)$$

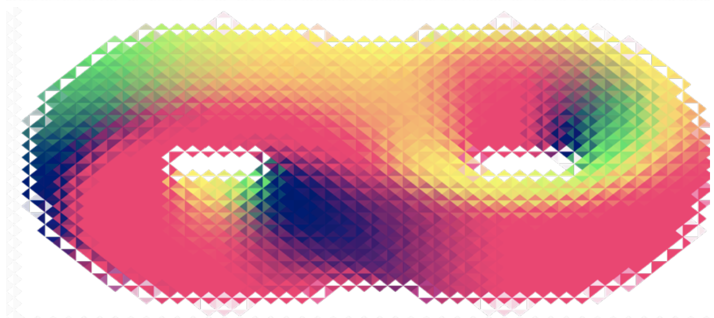
Quantum Curves

$N \to \infty$



$F_0(\mu)$

$O(1/N)$



$$\sum_{g \geq 0} N^{2-2g} F_g(\mu)$$

N finite

non-perturbative

Complex Curves in Phase Space

algebraic curve = level set

$$\Sigma : H(x,y) = 0$$

Hamilton-Jacobi theory

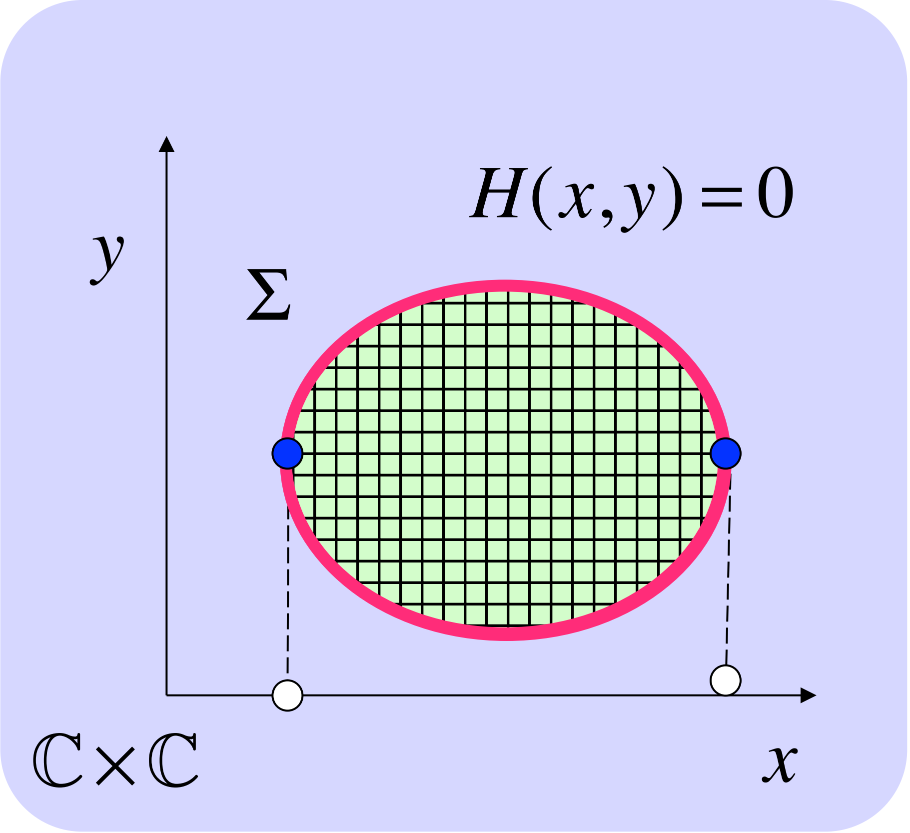
$$y = p(x)$$

branch pts = turning pts

Liouville form $\omega = y dx$

action

$$S(x) = \int_{x_*}^{x} \omega$$



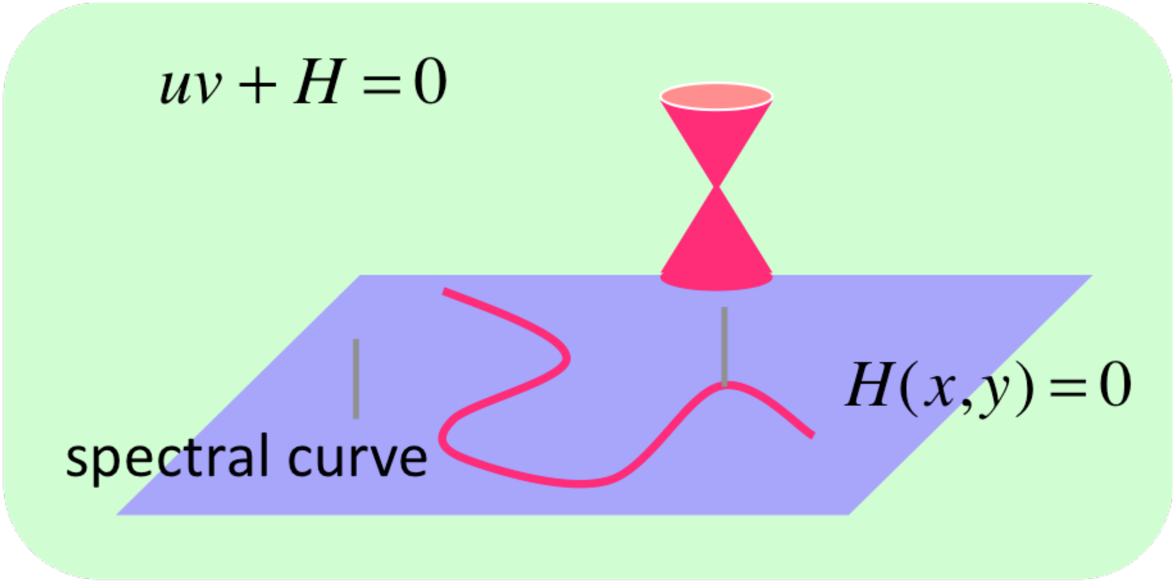
BS Quantization

$$dx \wedge dy \sim \hbar = g_s$$

Topological string on hypersurface CY_3

Calabi-Yau hypersurface in $\mathbb{C}^4$

$$X : uv + H(x,y) = 0$$



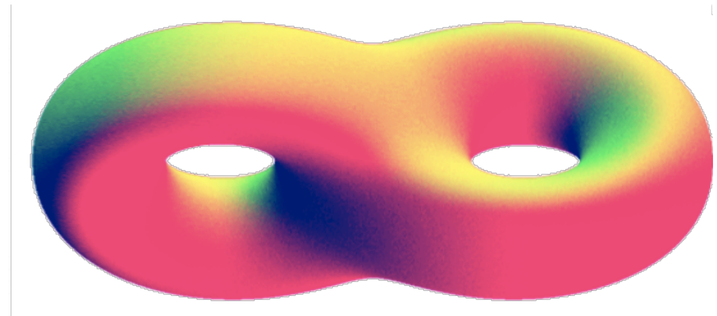
Period maps

$$\Omega = \frac{du}{u} \wedge dx \wedge dy \to \omega = y dx$$

Matrix models and chiral CFT

Eigenvalue density/matrix resolvent = collective boson field

$$\partial\varphi(x) = \mathrm{Tr}\frac{1}{x-\Phi}$$



$\varphi(x)$

Free scalar field

$$Z_{matrix} = \int D\varphi \cdot e^{-S[\varphi]}, \quad S = \int (\partial\varphi)^2 + O(1/g_s)$$

Fermions = D-branes

$$\psi(x) = e^{i\varphi(x)} = \det\left(x-\Phi\right)$$

$$\psi^*(x) = e^{-i\varphi(x)} = \det\left(x-\Phi\right)^{-1}$$

Loop Equations & Virasoro Constraints

Invariance under diffeomorphism $x \to \tilde{x}(x)$

Generated by stress-tensor

$$T(x) = \tfrac{1}{2}(\partial\varphi)^2(x)$$

Loop equations

$$\langle T(x) \rangle = W'(x)^2 + f(x)$$

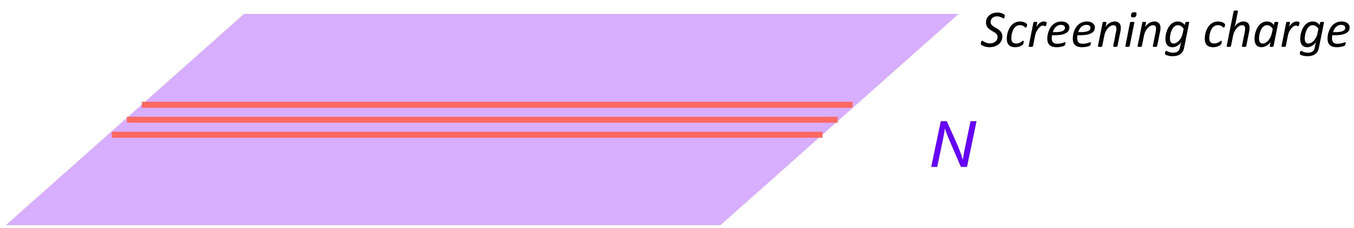
SU(2) affine KM algebra structure

Currents

$$J_+ = e^{2i\varphi} = \psi_2^* \psi_1$$
$$J_3 = \partial\varphi = \psi_1^* \psi_1 - \psi_2^* \psi_2$$
$$J_- = e^{-2i\varphi} = \psi_1^* \psi_2$$

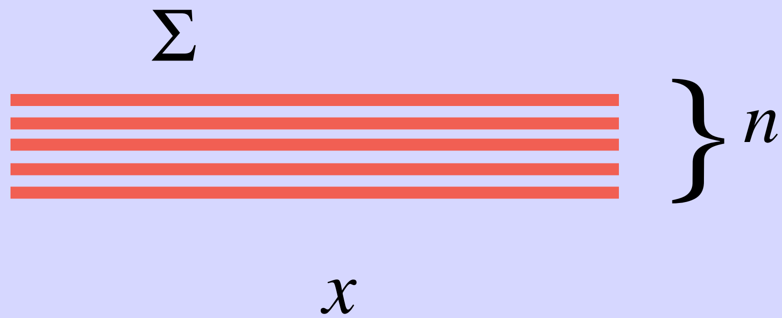
In the limit $W = 0$

$$Z_{matrix} = \left\langle \exp \int J_+ \right\rangle_N$$



Nilpotent Geometry

$$y^n = 0$$

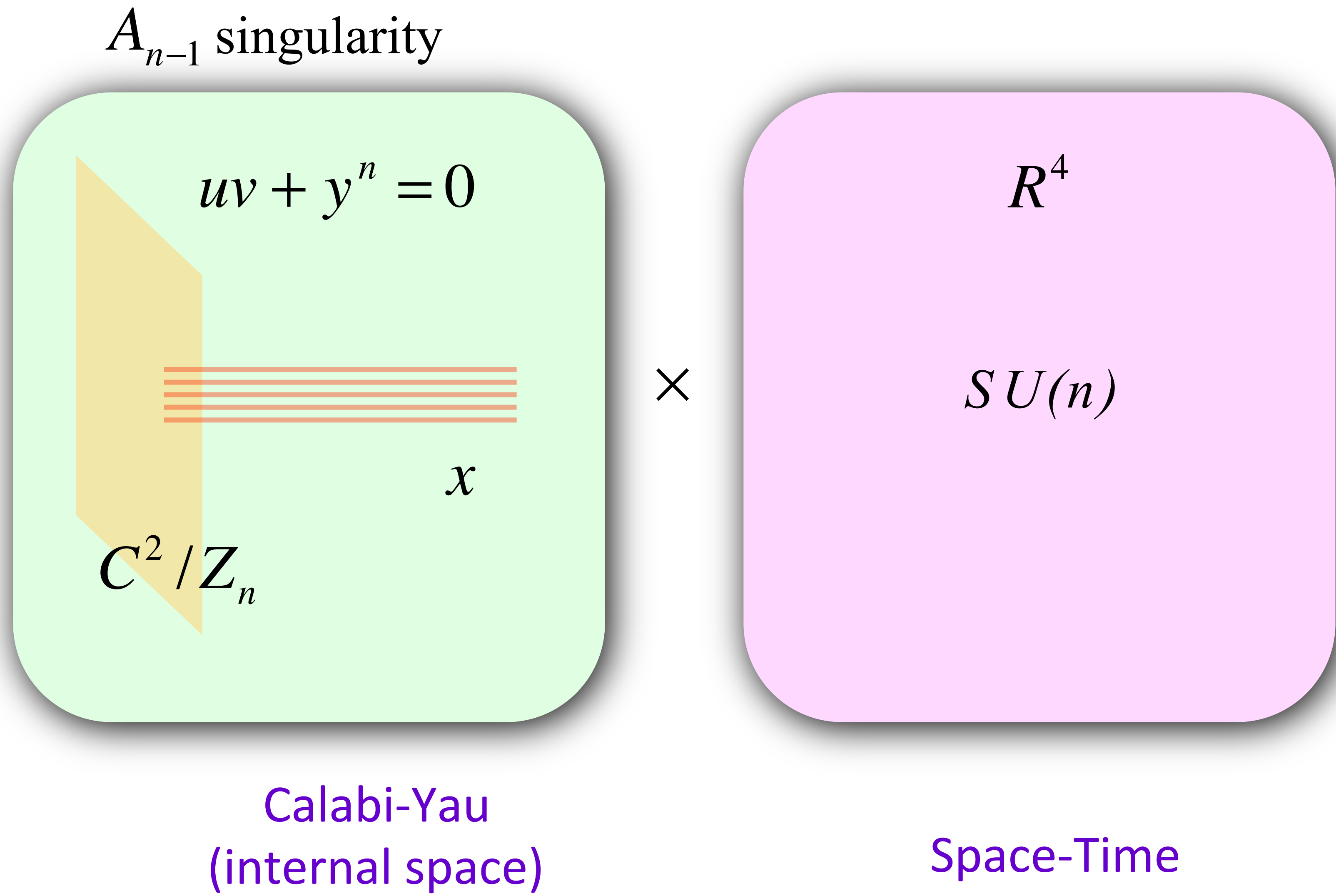


Nilpotent algebra

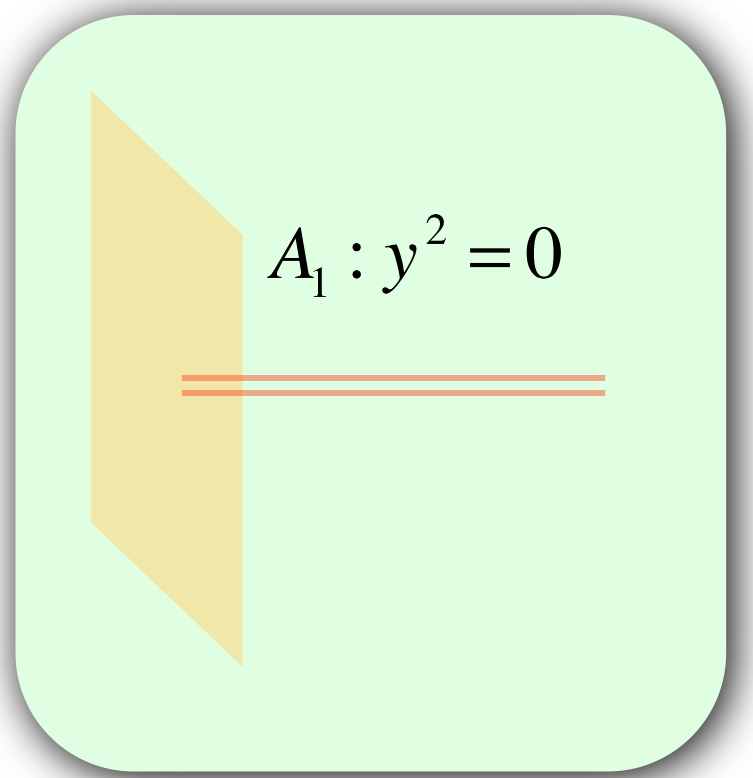
$$y^n = 0$$

$$\hat{y} = g_s \partial_x + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & 0 \end{pmatrix}$$

Geometry of SU(n) Gauge Theory

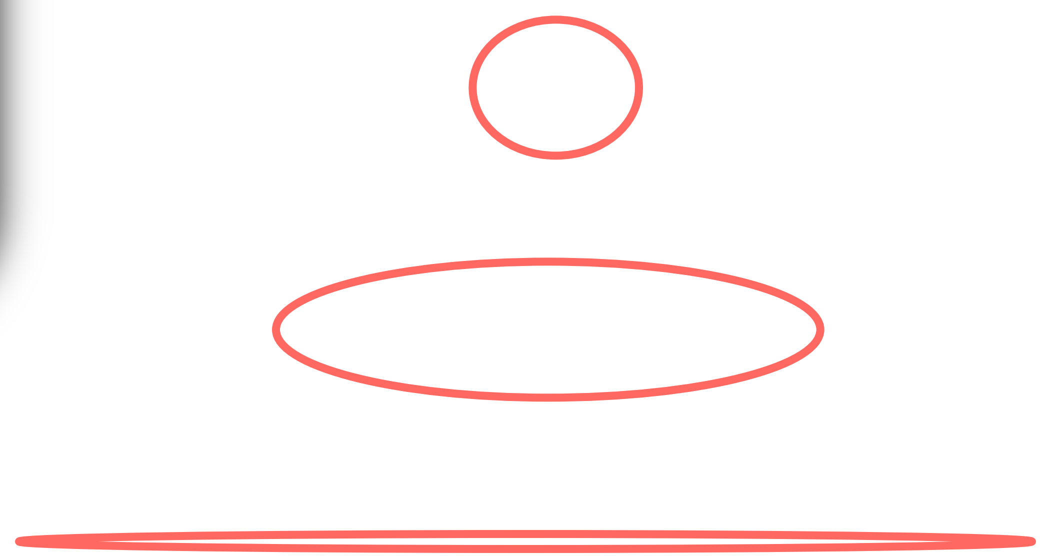


Singularities & Matrix Models



$$\int d\Phi \cdot e^{-a \mathrm{Tr}\,\Phi^2}$$

$$\lim_{a \to 0} \left\{ y^2 = a^2 (x^2 + \mu) \right\}$$



Matrix Models & Liouville Theory

$$Z_{matrix} = \int d^N x \cdot \prod (x_I - x_J)^2 \cdot e^{\sum W(x_I)/g_s}$$

CFT Coulomb gas correlation function

$$Z_{matrix} = \left\langle \int d^N x \cdot \prod e^{2i\varphi(x_i)} \cdot e^{\oint \partial\varphi \, W/g_s} \right\rangle_N$$

Screening charge

$$Z_{matrix} = \left\langle e^{\int e^{2i\varphi}} \cdot e^{\oint \partial\varphi \, W/g_s} \right\rangle_N$$

Large N :   Coulomb gas picture.

Beta Ensemble

General Liouville theory $b \neq i,\ \varepsilon_1 \neq -\varepsilon_2$

$$Z = \left\langle \exp \int e^{b\varphi} \right\rangle_N$$

Generalized matrix model measure

$$Z = \int d^N x \cdot \prod (x_I - x_J)^{2\beta} e^{\frac{\sqrt{\beta}}{g_s} \sum_I W(x_I)}, \quad \beta = -b^2$$

$$\beta = 1/2 : SO(N), \quad \beta = 2 : Sp(N)$$

Generalized loop equations

$$T(x) = \tfrac{1}{2} (\partial\varphi)^2 + Q \partial^2 \varphi$$

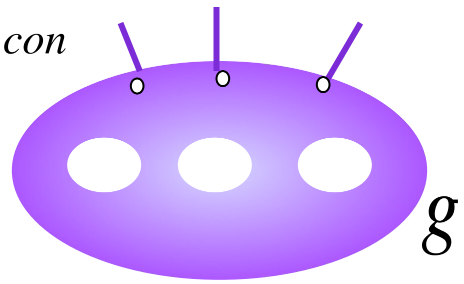
Eynard-Orantin: All-Genus Solution

Recursion relations for correlation functions, completely geometric

$$W(x_1,\cdots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1-\Phi}\cdots\mathrm{Tr}\frac{1}{x_n-\Phi}\right\rangle_{con}$$

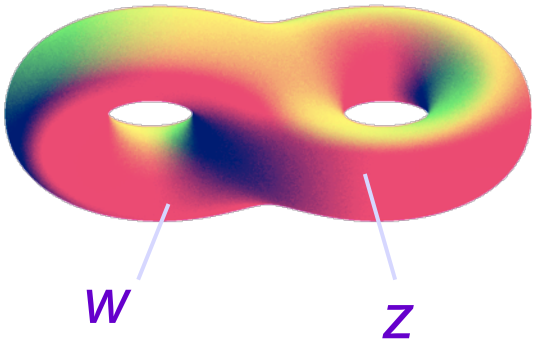
$$= \langle \partial\varphi(x_1)\cdots\partial\varphi(x_n)\rangle_{con}$$

$$= \sum_g g_s^{\,2g-2}\, W_g$$

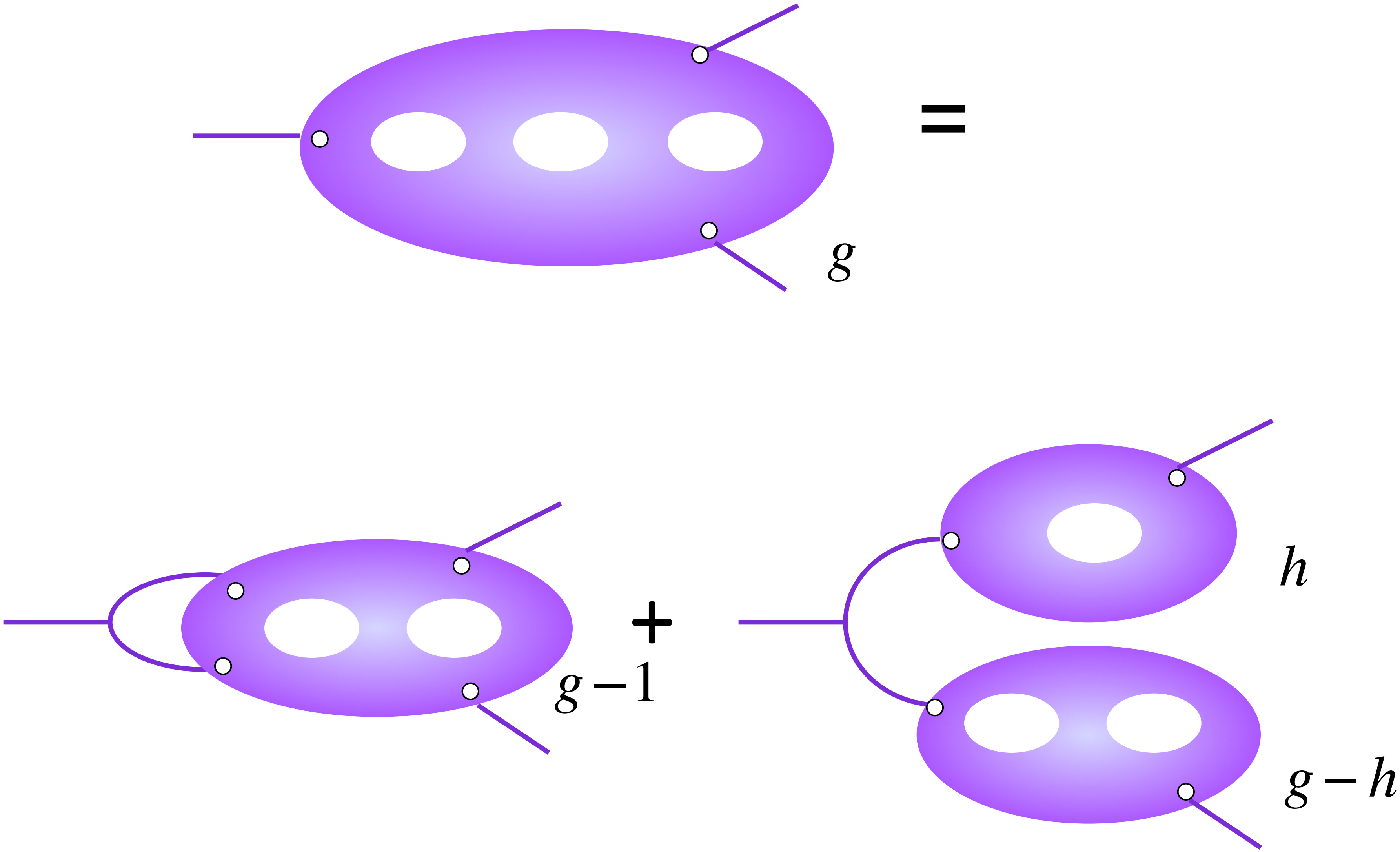


Lowers genus, adds more insertions. Reduces to Bergmann kernel

$$W_0(w,z) = B(w,z)dwdz$$

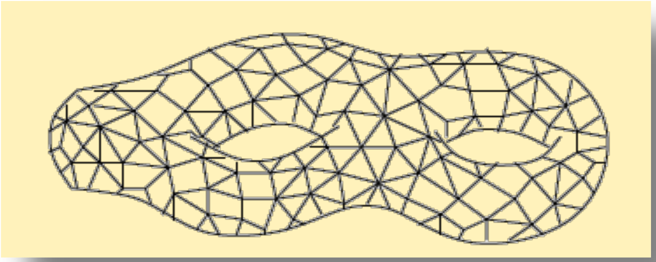


Recursion Relations



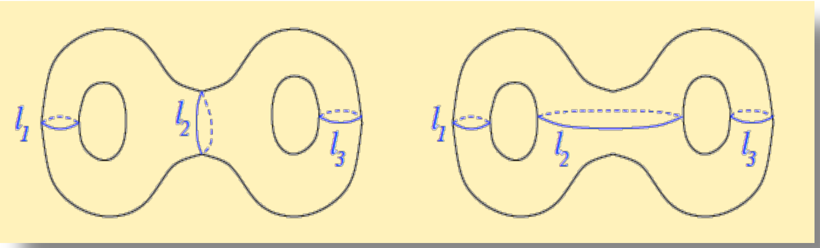
Quantum Spectral Curves

Random surfaces



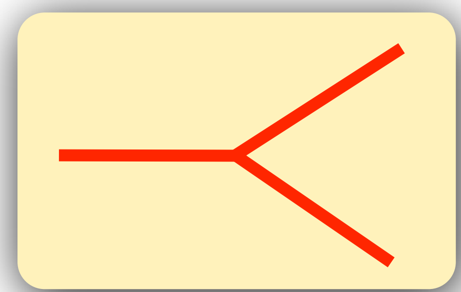
$$y^p = x^q$$

Geometry of $\mathcal{M}_g$



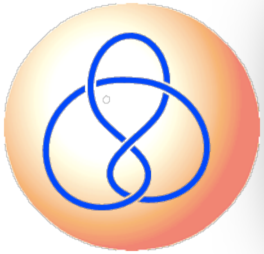
$$y = \sin\sqrt{x}$$

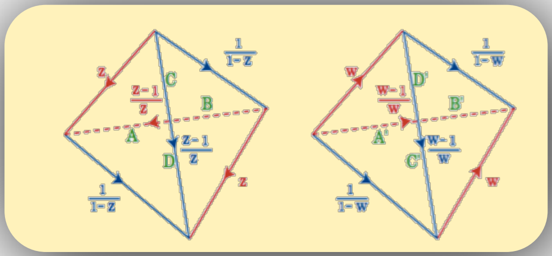
Toric Calabi-Yau



$$e^x + e^{-y} = 1$$

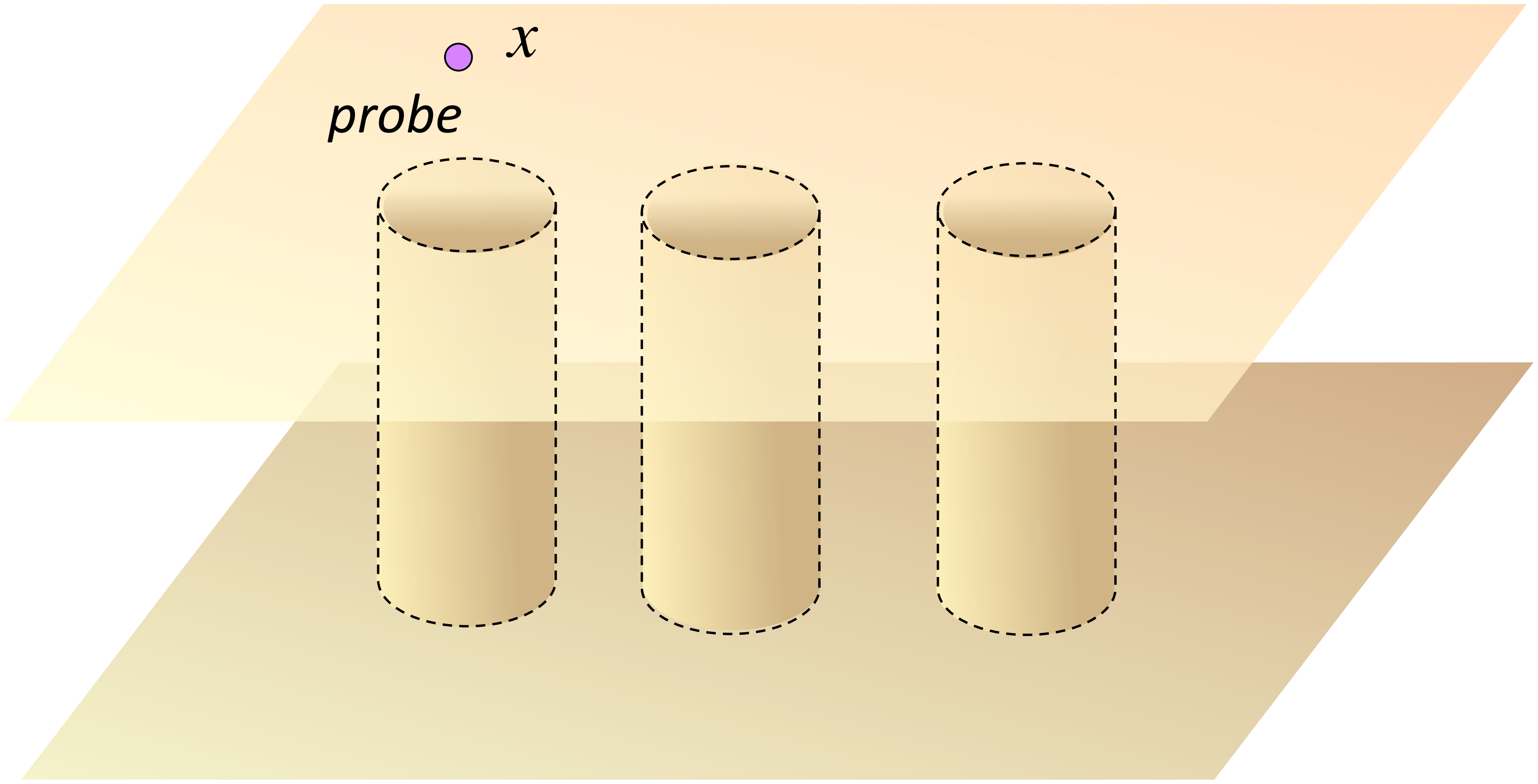
Hyperbolic knots



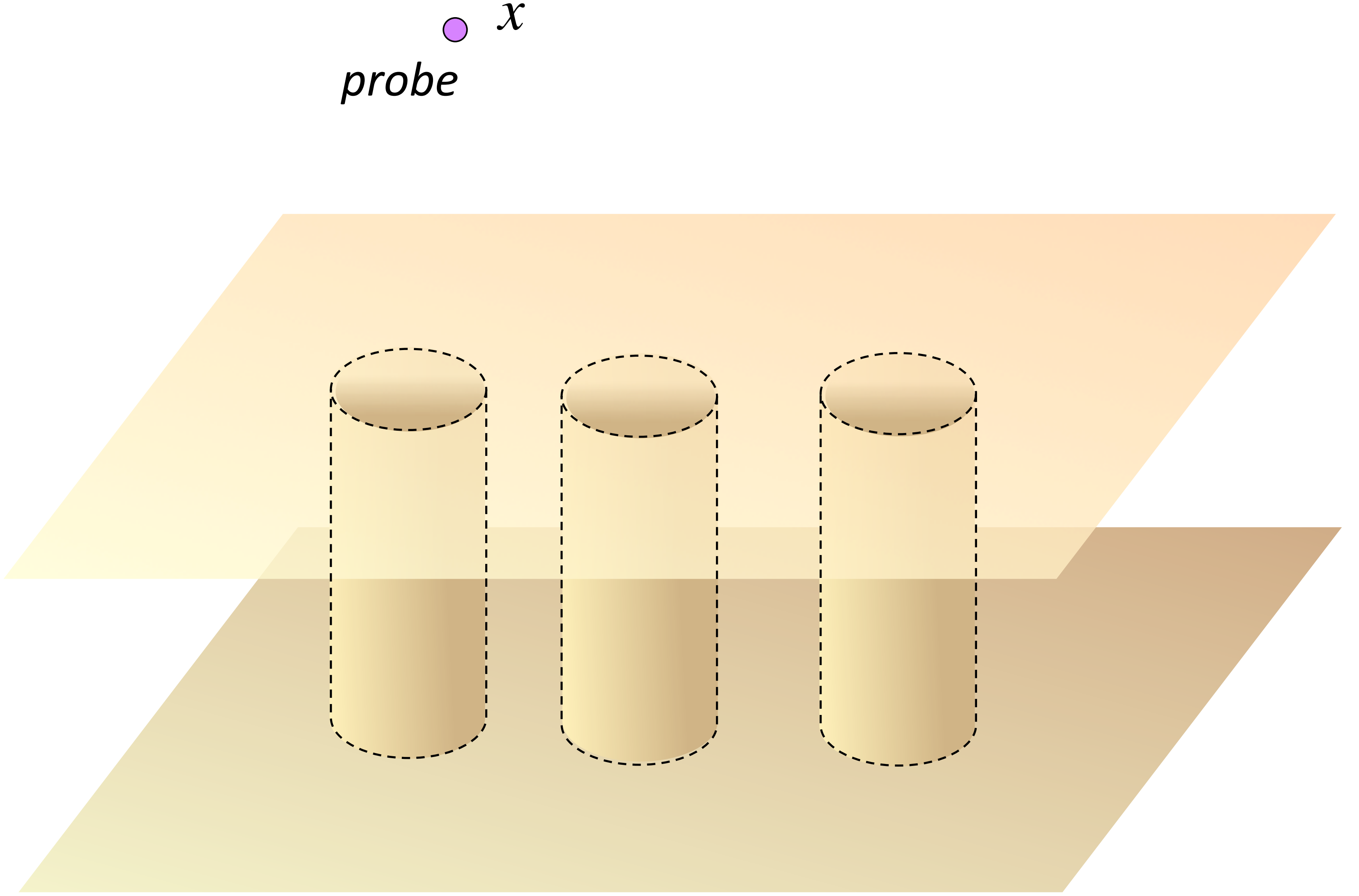


$$A(e^x, e^y) = 0$$

Effective Geometry

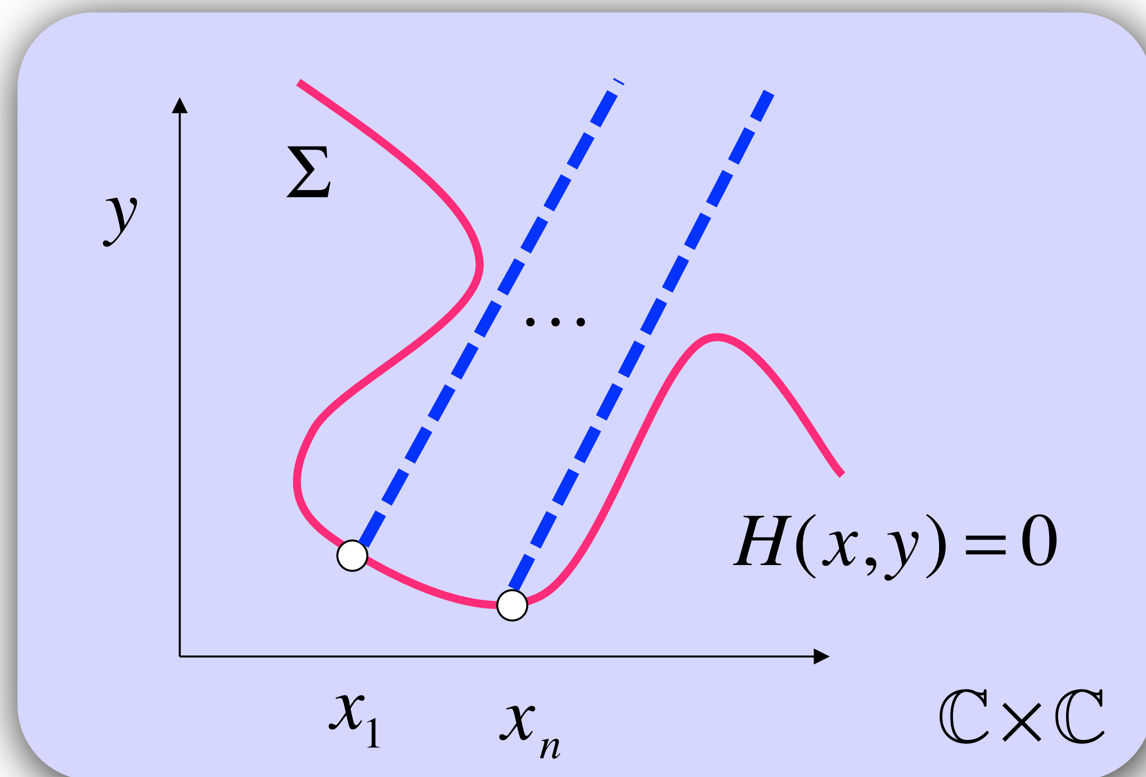


Are both sheets visible?



x
probe

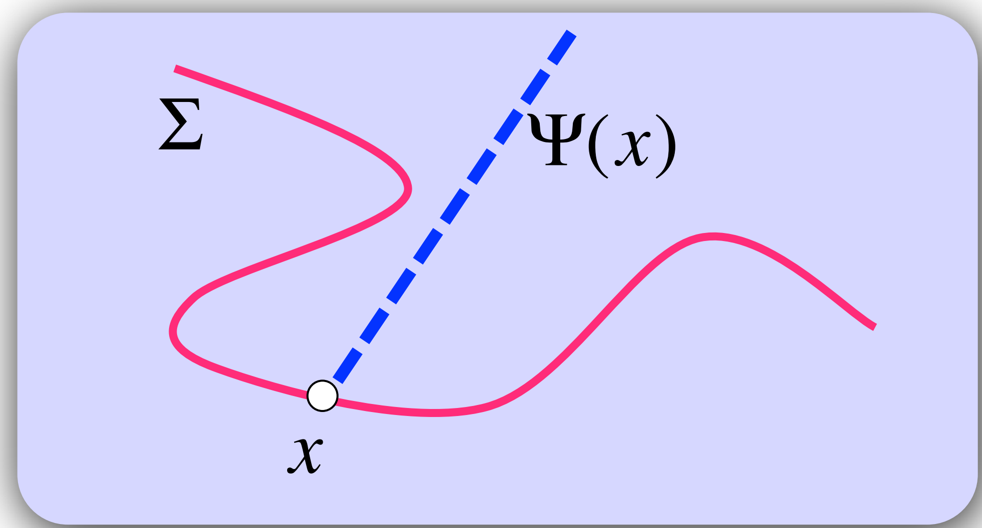
Topological D-Branes



Open string partition function

$$\Psi(x_1,\cdots,x_n) = \left\langle \psi(x_1)\cdots\psi(x_n) \right\rangle = \left\langle \prod_i \det(\Phi - x_i) \right\rangle$$

Quantum Wave Function



Single brane wave function $\Psi(x)$

$$\hat{H}\Psi = 0, \quad \hat{H} = \hat{H}(\hat{x}, \hat{y})$$

Quantization of phase space

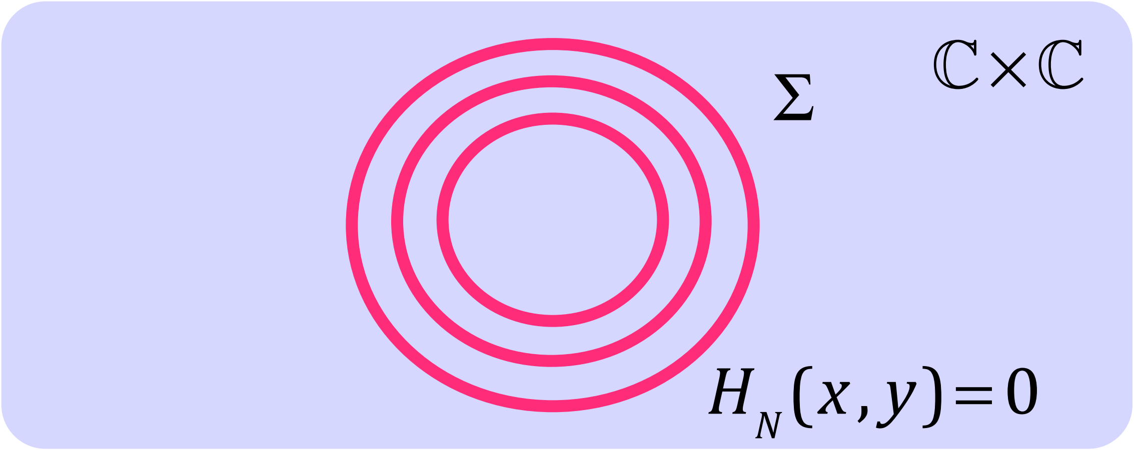
$$\hat{y} = -g_s \frac{\partial}{\partial x}, \quad [\hat{x}, \hat{y}] = g_s$$

Gaussian Matrix Model

$$\Psi_N(x) = \left\langle \det\left(\Phi - x\right) \right\rangle_N = H_{N-1}(x) \cdot e^{-x^2/2}$$

Eigenfunctions of harmonic oscillator

$$\left(-g_s^2 \frac{\partial^2}{\partial x^2} + x^2 - g_s(2N-1) \right) \Psi_N(x) = 0$$



Antibranes

$$\Psi^*_N(x) = \left\langle \frac{1}{\det(\Phi - x)} \right\rangle_N = \int \frac{\Psi_N(y)}{y-x}\,dy$$

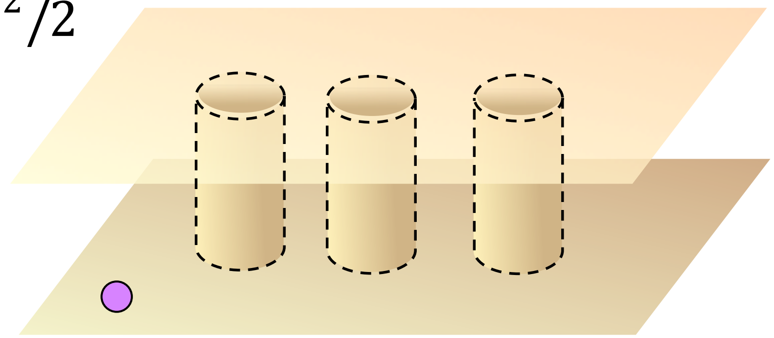
Non-normalizable eigenfunctions of harmonic oscillator

$$\widehat{H}_N \Psi^*_N(x) = 0$$

WKB behavior

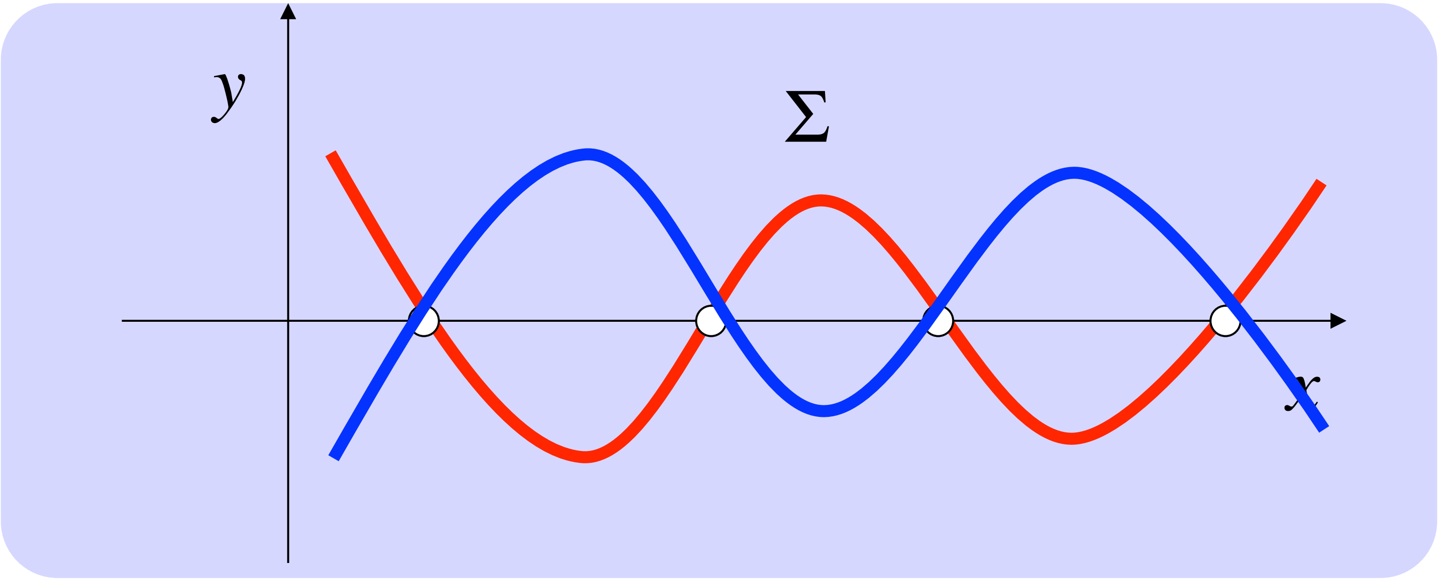
$$\Psi^*(x) \sim e^{+x^2/2}$$

Probe other sheet



Classical Spectral Curve

$$y^2 = W'(x)^2$$



branes: $y = W'(x)$

antibranes: $y = -W'(x)$

Super Gauge Theories

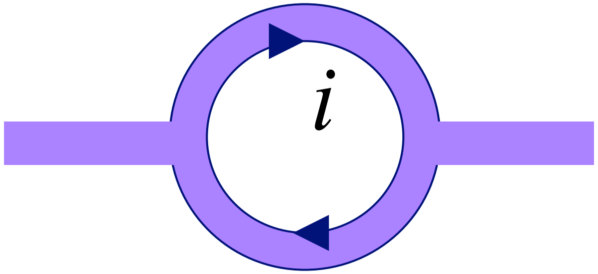
Super gauge group $U(N+k\,|\,k)$

Symmetries of $v=(z,\theta)\in\mathbb{C}^{N+k|k}$, $\quad |v|^2=\sum_{i=1}^{N+k}|z_i|^2+\sum_{a=1}^{k}\theta_a\bar{\theta}_a$

$$\Phi=\begin{pmatrix} A & B \\ C & D \end{pmatrix},\quad A,D \text{ even}, B,C \text{ odd.}$$

Large *N, M* expansion

$$\mathrm{Str}(1)=\sum_{i=1}^{N+k}1-\sum_{j=1}^{k}1=N$$



Perturbatively

$$U(N+k\,|\,k)\cong U(N)$$

Super Matrix Models

Supergroup

$$Z_{N+k|k} = \frac{1}{\mathrm{Vol}\, U(N+k\,|\,k)} \int d\Phi \cdot e^{\mathrm{Str}\, W(\Phi)/g_s}$$

Volume Lie supergroup vanishes: need regularization

$$\int d\theta \cdot 1 = 0$$

Diagonalize: eigenvalues & anti-eigenvalues

$$\Phi \sim \begin{pmatrix} x_i & 0 \\ 0 & y_j \end{pmatrix} \qquad S = \sum_{i=1}^{N+k} W(x_i) - \sum_{j=1}^{k} W(x_i)$$

Super Jacobian

$$d\Phi \;\rightarrow\; d^{N+k}x\, d^k y \frac{\prod_{i<j}\left(x_i - x_j\right)^2 \prod_{a<b}\left(y_a - y_b\right)^2}{\prod_{i,a}\left(x_i - y_b\right)^2}$$

Eigenvalues of charge +1 and -1 (Coulomb gas model)

Resolvent

$$ydx \sim \mathrm{Str}\frac{dx}{\Phi - x} = \sum_i \frac{dx}{x_i - x} - \sum_i \frac{dx}{y_a - x}$$

Same periods as U(N) model

Non-Perturbative Corrections

[Vafa] U(n+k|k) has more Casimirs than U(N), e.g.

$$\Delta = \prod_{i,a} \left(x_i - y_a \right)$$

General deformation

$$W = \sum_n t_n \mathrm{Str}\, \Phi^n$$

For example, for U(2|1)

$$\Delta = -\frac{1}{2} \left(\mathrm{Str}\, \Phi^2 - \left(\mathrm{Str}\, \Phi \right)^2 \right) = 0 \ \text{ for } \ U(1)$$

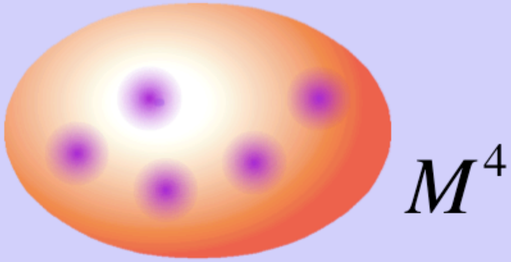
Non-Perturbative Corrections

So correlators $\left\langle \Delta \right\rangle_{N+k|k}$ are sensitive to *k.*

General pattern: calculation of partition function and correlation functions for U(N+k|k) depend non-perturbatively (order e^{-N}) on *k.*

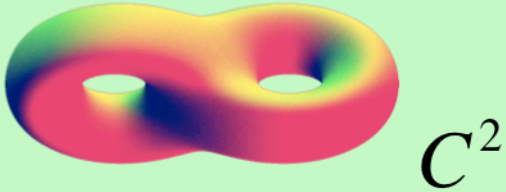
Working with supermatrix models seems to probe the full effective geometry.

Gauge Theories



self-dual connections
on 4-manifolds

Conformal Field Theory



2d QFT on algebraic curves
reps of infinite dim algebras

Matrix Models

$$\int_{N\times N} d\Phi \cdot e^{TrW(\Phi)}$$

Eigenvalue distribution

(Topological) Strings

holomorphic curves,
mirror symmetry